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Cutting techniques for mixed 0-1 problems
  Relaxation hierarchies
  Cut generation

Branch&Bound Based Outer Approximation
  Second Order Cone Subproblems
  Linear Outer Approximations
  Algorithm

Computational Results
Mixed 0-1 second order cone problem

\[
\min \quad c^T x \\
\text{s.t.} \quad Ax = b \\
\quad x \succeq 0 \\
\quad (x)_j \in \{0, 1\} \quad \forall j \in J \subseteq \{1, \ldots, n\},
\]

\[
x \succeq 0 : \\
x = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_{\text{noc}} \end{pmatrix}, \quad x_i \in \mathbb{R}^{k_i}, \quad \sum_{i=1}^{\text{noc}} k_i = n, \\
(x_i)_1 \geq \| (x_i)_{2:k_i} \|_2, \quad (i = 1, \ldots, \text{noc})
\]
Related sets

- binary feasible set $C^0 = \{ x \in \mathbb{R}^n : Ax = b, x \succeq 0, x_j \in \{0, 1\} \ \forall j \in J \}$,
Related sets

- binary feasible set $C^0 = \{x \in \mathbb{R}^n : Ax = b, x \succeq 0, x_j \in \{0, 1\} \quad \forall j \in J\}$,
- continuous relaxation $C = \{x \in \mathbb{R}^n : Ax = b, x \succeq 0, x_j \in [0, 1] \quad \forall j \in J\}$,
Related sets

- binary feasible set $C^0 = \{ x \in \mathbb{R}^n : Ax = b, x \succeq 0, x_j \in \{0, 1\} \; \forall j \in J \}$,
- continuous relaxation $C = \{ x \in \mathbb{R}^n : Ax = b, x \succeq 0, x_j \in [0, 1] \; \forall j \in J \}$,
- convex hull of the binary feasible set $\text{conv}(C^0)$
Hierarchies of relaxations

**Goal:** A tight convex relaxation of the binary feasible set $C^0$!

**Approach:** Lift-and-project-techniques: Stubbs and Mehrotra’s generalizations of hierarchies by Lovasz and Schrijver
Lifting and convex hulls

Describe \( \text{conv}(C \cap \{x_j \in \{0, 1\}\}) \) for a \( j \in J \) by **lifting**:

\[
\tilde{M}_j(C) = \left\{ (x, u_0, u_1, \lambda_0, \lambda_1) : \\
x = \lambda_0 u_0 + \lambda_1 u_1, \\
\lambda_0 + \lambda_1 = 1, \lambda_0, \lambda_1 \geq 0, \\
u_0 \in C, (u_0)_j = 0 \\
u_1 \in C, (u_1)_j = 1 \\
\right\}
\]
Lifting and convex hulls

Describe $\text{conv}(C \cap \{x_j \in \{0, 1\}\})$ for a $j \in J$ by lifting:

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Lifting and convex hulls

Describe $\text{conv}(C \cap \{x_j \in \{0, 1\}\})$ for a $j \in J$ by lifting:

$$\tilde{M}_j(C) = \left\{ (x, u_0, u_1, \lambda_0, \lambda_1) : \begin{align*}
    x &= \lambda_0 u_0 + \lambda_1 u_1, \\
    \lambda_0 + \lambda_1 &= 1, \lambda_0, \lambda_1 \geq 0, \\
    u_0 &\in C, \quad (u_0)_j = 0 \\
    u_1 &\in C, \quad (u_1)_j = 1
\end{align*} \right\},$$
Convex representation of $\tilde{M}_j(C)$

The substitution $\nu^0 := \lambda^0 u^0$ and $\nu^1 := \lambda^1 u^1$ can be used:

$$M_j(C) = \left\{ (x, \nu^0, \nu^1, \lambda^0, \lambda^1) : \begin{array}{l}
\nu^0 + \nu^1 = x \\
\lambda^0 + \lambda^1 = 1, \lambda^0, \lambda^1 \geq 0 \\
A\nu^0 - \lambda^0 b = 0, A\nu^1 - \lambda^1 b = 0 \\
v^0 \succeq 0, v^1 \succeq 0 \\
(\nu^0)_k \in [0, \lambda^0] \quad (k \in J, k \neq j) \\
(\nu^1)_k \in [0, \lambda^1] \quad (k \in J, k \neq j) \\
(\nu^0)_j = 0, (\nu^1)_j = \lambda^1 
\end{array} \right\}.$$
Convex representation of $\tilde{M}_j(C)$

The substitution $v^0 := \lambda^0 u^0$ and $v^1 := \lambda^1 u^1$ can be used:

$$M_j(C) = \left\{ (x, v^0, v^1, \lambda^0, \lambda^1) : \begin{array}{l}
    v^0 + v^1 = x \\
    \lambda^0 + \lambda^1 = 1, \lambda^0, \lambda^1 \geq 0 \\
    Av^0 - \lambda^0 b = 0, Av^1 - \lambda^1 b = 0 \\
    v^0 \succeq 0, v^1 \succeq 0 \\
    (v^0)_k \in [0, \lambda^0] \quad (k \in J, k \neq j) \\
    (v^1)_k \in [0, \lambda^1] \quad (k \in J, k \neq j) \\
    (v^0)_j = 0, (v^1)_j = \lambda^1
\end{array} \right\} ,$$

Projection on $x$:

$$P_j(C) := \{ x : (x, v^0, v^1, \lambda^0, \lambda^1) \in M_j(C) \}.$$
Higher order lifting

Make a **higher order lifting** for indices $B \subseteq J \rightsquigarrow M_B(C)$

**Projection** on $x$:

$$P_B(C) := \{ x : (x, (v^0_j, v^1_j, \lambda^0_j, \lambda^1_j) j \in B) \in M_B(C) \}.$$
Add an additional valid semidefinite constraint

Condition \( x_j = x_j^2 \) can be expressed as

\[
(*) \quad V_B^1 - x_B x_B^T = 0
\]

with \( V_B^1 := (v_B^{11}, \ldots, v_B^{1|B|}) \).
Add an additional valid semidefinite constraint

Condition \( x_j = x_j^2 \) can be expressed as

\[
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\]

with \( V_B^1 := (v_B^{11}, \ldots, v_B^{1|B|}) \).

Add convex relaxation of (*):

\[
M_B^+(C) = \{(x, (v_j^0, v_j^1, \lambda_j^0, \lambda_j^1)) \in B) \in M_B(C), \quad V_B^1 - x_B x_B^T \succeq_{sd} 0 \},
\]
Add an additional valid semidefinite constraint

Condition \( x_j = x_j^2 \) can be expressed as

\[
(*) \quad V^1_B - x_B x_B^T = 0
\]

with \( V^1_B := (v^1_{B_1}, \cdots v^1_{|B|}) \).

Add convex relaxation of (*):

\[
M_B^+(C) = \left\{ (x, (v^{0j}, v^{1j}, \lambda^{0j}, \lambda^{1j}) j \in B) \in M_B(C), \ V^1_B - x_B x_B^T \succeq_{sd} 0 \right\},
\]

Projection on \( x \):

\[
P_B^+(C) := \left\{ x : (x, (v^{0j}, v^{1j}, \lambda^{0j}, \lambda^{1j}) j \in B) \in M_B^+(C) \right\},
\]
Convex hulls and tighter relaxations

Theorem [Stubbs, Mehrotra ’99]
Let $B \subseteq J = \{i_1, \ldots, i_p\}$.

It holds that

$P_J(C) = \text{conv}(C \cap \{x_j \in \{0, 1\} \}) \subseteq \text{conv}(C_0) \subseteq P_B(C) \subseteq P_B(C)$
Convex hulls and tighter relaxations

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Convex hulls and tighter relaxations

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It holds that

\[ P_j(C) = \text{conv}(C \cap \{x_j \in \{0, 1\}\}) \subseteq \text{conv}(C^0) \subseteq P_B^+(C) \subseteq P_B(C) \]
**Goal:** Find convex inequalities

\[ x^T Q x + \alpha^T x \geq \beta, \]

valid for all \( x \in C^0 \).
Theorem (Cezik, Iyengar ’05, D. ’08)

Let \( \text{int}(\text{conv}(C^0)) \neq \emptyset \), \( B \subseteq J \), \( V_B^1 := (v_B^{11}, \cdots, v_B^{1|B|}) \), then

\[
Q \bullet V_B^1 + \alpha^T x \geq \beta \quad \text{with} \quad Q = Q^T = (q^1, \ldots, q^{|B|})
\]

is valid for all \((x, (v^{1k}, v^{0k}, \lambda^{1k}, \lambda^{0k}) \forall k \in B) \in M^+_B(C)\)
Dual cuts

Theorem (Cezik, Iyengar ’05, D. ’08)

Let \( \text{int}(\text{conv}(C^0)) \neq \emptyset \), \( B \subseteq J \), \( V_B^1 := (v_{B}^{11}, \ldots, v_{B}^{1|B|}) \), then

\[
Q \cdot V_B^1 + \alpha^T x \geq \beta \quad \text{with} \quad Q = Q^T = (q^1, \ldots, q^{|B|})
\]
is valid for all \( (x, (v^{1k}, v^{0k}, \lambda^{1k}, \lambda^{0k}) \forall k \in B) \in M_B^+(C) \) if and only if there exist

\[
y^k = (y^{1,k}, y^{2,k}, y^{3,k}, y^{4,k}, y^{5,k}, y^{6,k}) \in \mathbb{R}^{n+2+3m}
\]
\[
s^k = (s_{v^0}, s_{v^1}, s_{\lambda^0}, s_{\lambda^1}) \in \mathbb{R}^{2n+2},
\]
\[
s_x \in \mathbb{R}^n, \quad S^6 = (s^{6,1}, \ldots, s^{6,|B|+1}) \in \mathbb{R}^{|B|+1, |B|+1}.
\]
\[ - \sum_{k=1}^{\vert B \vert} y^{1,k} + \left( e_{i_1}, \ldots, e_{i_{\vert B \vert}} \right) \begin{pmatrix} s^{6,1}_{\vert B \vert+1} \\ \vdots \\ s^{6,\vert B \vert}_{\vert B \vert+1} \end{pmatrix} + \left( e_{i_1}, \ldots, e_{i_{\vert B \vert}} \right) s^{6,\vert B \vert+1}_{1:\vert B \vert} + s_x = \alpha, s_x \geq 0, \]

\[ I^n y^{1,k} + y^3_k e_{i_k} + A^T y^{5,k} + s_{v,k0} = 0, s_{v,k0} \geq 0, \]

\[ I^n y^{1,k} + y^4_k e_{i_k} + A^T y^{6,k} + \tilde{E} y^{7,k} + \left( e_{i_1}, \ldots, e_{i_{\vert B \vert}}, 0 \right) s^{6,k} + s_{v,k1} = q^k, s_{v,k1} \geq 0, \]

\[ y^2_k - b^T y^{5,k} + s_{\lambda,k0} = 0, s_{\lambda,k0} \geq 0, \]

\[ y^2_k - y^4_k - b^T y^{6,k} + s_{\lambda,k1} = 0, s_{\lambda,k1} \geq 0, \quad \forall k \in B, \]

\[ S^6 \succeq_{sd} 0, \]

\[ \sum_{k=1}^{\vert B \vert} y^2_k - s^{6,\vert B \vert+1}_{\vert B \vert+1} = \beta. \]
Lemma (Cezik, Iyengar ’05)

The convex quadratic inequality $x_B^T Q x_B + \alpha^T x \geq \beta$, with $-Q \succeq_{sd} 0$ is valid for $P_B^+(C)$, if $(Q, \alpha, \beta)$ satisfy (1-7).

Proof:

$x_B^T Q x_B \geq V_B^1 \cdot Q$ holds for $M_B^+(C)$
Lemma

a) The inequality $\alpha^T x \geq \beta$ is valid for $P_B(C)$, if there exists $(Q = 0, \alpha, \beta)$, satisfying (1) - (5) with $S^6 = 0$ and (7).
Lemma

a) The inequality $\alpha^T x \geq \beta$ is valid for $P_B(C)$, if there exists $(Q = 0, \alpha, \beta)$, satisfying (1) - (5) with $S^6 = 0$ and (7).

b) The convex quadratic inequality $x_B^T Q x_B + \alpha^T x \geq \beta$, $Q = \text{diag}(q_1, \ldots, q_{|B|})$, with $q_i \leq 0$ is valid for $P_B(C)$, if $(Q, \alpha, \beta)$ satisfy (1-5) with $S^6 = 0$ and (7).

Proof: $x_B^T Q x_B \geq V^1_B \bullet Q$ holds for $M_B(C)$.
Let $\bar{x} \in C$ be a fractional solution.

$$\min \quad \alpha^T \bar{x} - \beta$$

s.t. \quad (0, \alpha, \beta) \text{ satisfy (1)-(5), (7)}

\quad S^6 = 0
Linear cut $\alpha^T x - \beta \geq 0$ generating SOCP

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$$\min \quad \alpha^T \bar{x} - \beta$$

s.t.  \quad (0, \alpha, \beta) \text{ satisfy (1)-(5), (7)}

$S^6 = 0$
Diagonal quadratic cut \( x_B^T Q x_B + \alpha^T x - \beta \geq 0 \) generating SOCP

Let \( \bar{x} \in C \) be a fractional solution.

\[
\begin{align*}
\min \quad & \bar{x}_B^T Q \bar{x}_B + \alpha^T \bar{x} - \beta \\
\text{s.t.} \quad & (Q, \alpha, \beta) \text{ satisfy (1)-(5), (7)} \\
& S^6 = 0 \\
& Q = diag(q_1, \ldots q_l) \leq 0
\end{align*}
\]
Diagonal quadratic cut \( x_B^T Q x_B + \alpha^T x - \beta \geq 0 \)

generating SOCP

Let \( \bar{x} \in C \) be a fractional solution.

\[
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\text{s.t.} \quad & (Q, \alpha, \beta) \text{ satisfy } (1)-(5), (7) \\
& S^6 = 0 \\
& Q = \text{diag}(q_1, \ldots, q_l) \leq 0
\end{align*}
\]
Quadratic cut $x_B^T Q x_B + \alpha^T x - \beta \geq 0$ generating semidefinite Problem

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$$\text{s.t. } (Q, \alpha, \beta) \text{ satisfy (1)-(7)}$$
$$-Q \succeq_{sd} 0$$
Quadratic cut $x_B^T Q x_B + \alpha^T x - \beta \geq 0$ generating semidefinite Problem

Let $\bar{x} \in C$ be a fractional solution.

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s.t. $(Q, \alpha, \beta)$ satisfy (1)-(7)

$-Q \succeq_{sd} 0$
Proposition (Subgradient cuts for SOCP)

Let $\bar{x} \notin P_B(C)$. And $\hat{x}$ be the optimal solution of the minimum distance problem

$$\min_{x \in P_B(C)} \|x - \bar{x}\|_2,$$

then

$$(\hat{x} - \bar{x})^T x \geq (\hat{x} - \bar{x})^T \hat{x}$$

is a valid linear inequality for $x \in P_B(C)$ that cuts off $\bar{x}$. 

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Proposition (Subgradient cuts for SOCP)

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\[
(\hat{x} - \bar{x})^T x \geq (\hat{x} - \bar{x})^T \hat{x}
\]

is a valid linear inequality for \( x \in P_B(C) \) that cuts off \( \bar{x} \).
**Proof:** Follows from result by Stubbs, Mehrotra (’99). □
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Computational Results
Mixed Integer Second Order Cone Problem

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
(MISOCPP) & \quad g_i(x) \leq 0, \ (i = 1, \ldots, noc), \\
& \quad x_J \in \mathbb{Z}^{|J|} \cap [l, u],
\end{align*}
\]

where \( g_i(x) := -(x_i)_1 + \|(x_i)_{2:k_i}\|_2 \). Remember \( x = \begin{pmatrix} x_1 \\ \vdots \\ x_{noc} \end{pmatrix}, \quad x_i \in \mathbb{R}^{k_i}, \quad \sum_{i=1}^{noc} k_i = n. \)

The function \( g_i : \mathbb{R}^n \rightarrow \mathbb{R} \) is

- differentiable on \( \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : \|(x_i)_{2:k_i}\|_2 = 0\} \)
- subdifferentiable on \( \{x \in \mathbb{R}^n : \|(x_i)_{2:k_i}\|_2 = 0\} \).
Assumptions:

A1 The feasible set is bounded
A2 Slater CQ holds at every occurring SOCP

$\Rightarrow$ KKT optimality conditions and primal dual optimality in every occurring SOCP!
Let $x^k_J$ be an integer assignment.

$$\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
\text{NLP}(x^k_J) & \quad g_i(x) \leq 0, \quad (i = 1, \ldots noc), \\
& \quad x_J = x^k_J.
\end{align*}$$
Nonlinear Subproblems

Let $x_j^k$ be an integer assignment.

$$\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
\text{NLP}(x_j^k) & \quad g_i(x) \leq 0, \quad (i = 1, \ldots, noc), \\
& \quad x_J = x_j^k.
\end{align*}$$

Let $\bar{x}$ solve $\text{NLP}(x_j^k)$. A2 $\Rightarrow$ There exist $\mu$, $\lambda \geq 0$ and $\xi_i \in \partial g_i(\bar{x})$:

$$(KKT1) \quad c + (A^T, I_J^T)\mu + \sum_{i:g_i(\bar{x})=0} \lambda_i \xi_i = 0,$$

where $I_Jx = x_J$. 
Feasibility problems

If NLP($x_j^k$) is infeasible, then we solve the Feasibility Problem

\[
\begin{align*}
\text{min} & \quad u \\
\text{s.t.} & \quad Ax = b \\
F(x_j^k) & \quad g_i(x) \leq u, \quad (i = 1, \ldots, noc), \\
& \quad x_J = x_J^k, \\
& \quad u \geq 0.
\end{align*}
\]
If NLP($x_j^k$) is infeasible, then we solve the Feasibility Problem

$$\begin{align*}
\min & \quad u \\
\text{s.t.} & \quad Ax = b \\
F(x_j^k) & \quad g_i(x) \leq u, \quad (i = 1, \ldots, noc), \\
& \quad x_J = x_J^k, \\
& \quad u \geq 0.
\end{align*}$$

Let $(\bar{u}, \bar{x})$ solve $F(x_j^k)$ with $\bar{u} > 0$. $\Rightarrow$ There exist $\mu, \lambda \geq 0$ and $\xi_i \in \partial g_i(\bar{x})$:

(KKT2)

$$\begin{align*}
(A^T, I_J^T)\mu + \sum_{i:g_i(\bar{x})=\bar{u}} \lambda_i \xi_i &= 0, \\
\sum_{i:g_i(\bar{x})=\bar{u}} \lambda_i &= 1.
\end{align*}$$
Subgradient inequalities

Since $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex,

$$g_i(\bar{x}) + \xi^T (x - \bar{x}) \leq g_i(x)$$

holds for all $\xi \in \partial g_i(\bar{x})$, $\bar{x} \in \mathbb{R}^n$. 
Subgradient inequalities

Since \( g_i : \mathbb{R}^n \mapsto \mathbb{R} \) is convex,

\[
g_i(\bar{x}) + \xi^T (x - \bar{x}) \leq g_i(x)
\]

holds for all \( \xi \in \partial g_i(\bar{x}) \), \( \bar{x} \in \mathbb{R}^n \). \( \Rightarrow \)

The linearized constraint

\[
\text{(LC)} \quad g_i(\bar{x}) + \xi^T (x - \bar{x}) \leq 0
\]

provides outer approximation of \( g_i(x) \leq 0 \).
Subgradient inequalities

Since \( g_i : \mathbb{R}^n \mapsto \mathbb{R} \) is convex,

\[
g_i(\bar{x}) + \xi^T(x - \bar{x}) \leq g_i(x)
\]

holds for all \( \xi \in \partial g_i(\bar{x}), \bar{x} \in \mathbb{R}^n \). \( \Rightarrow \)

The linearized constraint

\[
(\text{LC}) \quad g_i(\bar{x}) + \xi^T(x - \bar{x}) \leq 0
\]

provides outer approximation of \( g_i(x) \leq 0 \).

Idea:
- Identify subgradients that satisfy KKT1 / KKT2.
- Use these subgradients to create linearized constraints (LC).
**Subgradients of NLP($x_j^k$)**

**Lemma 1**

*Assume A1 and A2 hold. Let $(\bar{x}, \bar{s}, \bar{y})$ be the primal-dual solution of NLP($x_j^k$).*

*Then there exist Lagrange multipliers* $\bar{\mu} = -\bar{y}, \bar{\lambda}_i = (\bar{s}_i)_1 \geq 0 \ (i \in A_{NLP} := \{i : g_i(\bar{x}) = 0\})$ *

*solving the KKT conditions (KKT1) in $\bar{x}$ with subgradients* $\xi_i(\bar{x}) = (0, \ldots, 0, \bar{\xi}_i^T, 0, \ldots, 0)^T$,  

*where*

\[
\begin{align*}
\bar{\xi}_i &= \nabla g_i(\bar{x}_i) \quad (i \in A_{NLP} : \|\bar{x}_i\| > 0), \\
\bar{\xi}_i &= -\frac{1}{(\bar{s}_i)_1} \bar{s}_i, \quad \text{if } (\bar{s}_i)_1 > 0 \\
\bar{\xi}_i &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \text{if } (\bar{s}_i)_1 = 0 \quad (i \in A_{NLP} : \|\bar{x}_i\| = 0).
\end{align*}
\]
**Subgradients of** $F(x_j^k)$

**Lemma 2**

Assume $A_1$ and $A_2$ hold. Let $(\bar{x}, \bar{u})$ solve $F(x_j^k)$ with $\bar{u} > 0$. Let $(\bar{s}, \bar{y})$ be the dual solution of $F(x_j^k)$. Then there exist Lagrange multipliers $\bar{\mu} = -\bar{y}$, $\bar{\lambda}_i = (\bar{s}_i)_1 \geq 0$ ($i \in A_F := \{i : g_i(\bar{x}) = \bar{u}\}$)

solving the KKT conditions (KKT2) in $(\bar{x}, \bar{u})$ with subgradients

$$\xi_i(\bar{x}) = (0, ... 0, \bar{\xi}_i^T, 0, ... 0)^T,$$

where

$$\begin{aligned}
\bar{\xi}_i &= \nabla g_i(\bar{x}_i) & (i \in A_F : \|(\bar{x}_i)_{2:k_i}\| > 0), \\
\bar{\xi}_i &= -\frac{1}{(\bar{s}_i)_1} \bar{s}_i, & \text{if } (\bar{s}_i)_1 > 0 \\
\bar{\xi}_i &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}, & \text{if } (\bar{s}_i)_1 = 0 \quad (i \in A_F : \|(\bar{x}_i)_{2:k_i}\| = 0).
\end{aligned}$$ (Sub2)
Linear Outer Approximations

$T$: set contains solutions of problems $NLP(x_j^k)$,
$S$: set contains solutions of problems $F(x_j^k)$.

$$\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad \xi_i(\bar{x})^T (x - \bar{x}) \leq 0, \quad \xi_i(\bar{x}) \text{ from Lemma 1, } \forall \bar{x} \in T, \\
& \quad \xi_i(\bar{x})^T (x - \bar{x}) \leq 0, \quad \xi_i(\bar{x}) \text{ from Lemma 2, } \forall \bar{x} \in S, \\
& \quad c^T x < c^T \bar{x}, \\
& \quad x_J \in \mathbb{Z}^{|J|} \cap [l, u].
\end{align*}$$

OA(T,S)

with continuous relaxation $OA^k(T,S)$. 
Branch & Bound Based Outer Approximation

\[ CUB := \infty, \text{Nodes} := \{ N^0 = (lb, ub) = (l, u) \}, \]

solve \( NLP([l, u]) \): if \( \bar{x} \in \mathbb{Z}^{|J|} \): STOP,

else \( S = \emptyset, T = \{ \bar{x} \} \).

while \( \text{Nodes} \neq \emptyset \): Select node \( N^k \), \( \text{Nodes} := \text{Nodes} \setminus N^k \),

1. Solve LP \( OA^k(T, S) \): solution \( x^k \).
   while \( (x^k_j \text{ integer}) \) \& \( (OA^k(T, S) \text{ feasible}) \)
   if \( (NLP(x^k_j) \text{ feasible}) \) \( T := T \cup \{ \bar{x} \}, CUB = \min\{CUB, c^T\bar{x}\} \)
   else compute solution \( \bar{x} \) of \( F(x^k_j) \), \( S := S \cup \{ \bar{x} \} \)
   compute solution \( x^k \) of updated \( OA^k(T, S) \)

2. if \( (OA^k(T, S) \text{ feasible}) \) branch on \( x^k_j \not\in \mathbb{Z} \):
   Create 2 nodes with bounds \( ub_j = \lfloor x^k_j \rfloor / lb_j = \lceil x^k_j \rceil \).
Convergence

Theorem
Assume A1 and A2 hold. Then the outer approximation algorithm terminates in a finite number of steps at an optimal solution of (MISOC) or with the indication, that it is infeasible.

Proof:
- A1, A2 ⇒ KKT and primal-dual optimality
- Subgradients in $OA(T, S)$ satisfy (KKT1) and (KKT2)

It can be shown: No integer assignment is generated twice.
□
Cutting techniques for mixed 0-1 problems
  Relaxation hierarchies
  Cut generation

Branch&Bound Based Outer Approximation
  Second Order Cone Subproblems
  Linear Outer Approximations
  Algorithm

Computational Results
Implementation

**SOCP Solver:** [Ulbrich, Hess, Drewes] (C-implementation)
- Infeasible primal-dual interior point approach based on (Tsuchiya, 98)

**B&C [Hess, Drewes]/ OA -Framework [Drewes]:** (C++-implementation)
- **Branching:** first / most fractional / combined fractional / pseudocost
- **Node selection:** depth first search / best bound first

**Semidefinite cut problems:** SeDuMi.

**LP/MIP-Solver:** CPLEX 10.01

**OA(Hybrid version):** solve NLP-relaxation every 10 nodes
Test problems

- 21 mixed 0-1 SOCP test problems (academic, ESTP and balancing problems),
- Dimensions $n = 5 \ldots 369$, $m = 3, \ldots 357$, $|J| = 1, \ldots 56$,
- Combinations of 2 node selections and 3 – 4 branching rules $\Rightarrow$ 138 test instances.
- Test with branch&bound and branch&cut in combination with each cut separately ($|B| = 2$; one cut per node)
- Test with branch&bound based outer approximation without cuts, and in combination with each cut separately ($|B| = 2$; one cut per node)
Comparison B&C and B&B-OA

B&B
B&B-OA
B&C- Linear
B&C-OA- Linear
B&C- SOC Quad
B&C-OA- SOC Quad
B&C- SDP Quad
B&C-OA- SDP Quad
B&C-Subgrad
B&C-OA- Subgrad

NLP-Nodes
LP-Nodes
Branch&Cut

B&C: Reductions by Cuts

B&C: Minimal Reductions by Cuts

B&C: Fraction of problems reduced by Cuts
Branch&Bound based Outer Approximation

**B&B-OA: Reductions by Cuts**

- B&C-OA-Linear
- B&C-OA-SOC Quad
- B&C-OA-SDP Quad
- B&C-OA-Subgrad

**B&B-OA: Minimal Reductions by Cuts**

- B&C-OA-Linear
- B&C-OA-SOC Quad
- B&C-OA-SDP Quad
- B&C-OA-Subgrad

**B&B-OA: Fraction of problems reduced by Cuts**

- B&C-OA-Linear
- B&C-OA-SOC Quad
- B&C-OA-SDP Quad
- B&C-OA-Subgrad
Conclusions

▶ Cuts reduce the B&B search trees for most of the test instances.
▶ In B&C: SOCP based quadratic cuts perform best.
▶ Cuts induce minor reductions of the B& B based OA search trees.
▶ In B& B - OA: subgradient and SDP based quadratic cuts perform best.
▶ SDP quadratic cuts are based on tightest relaxation
  \(\rightsquigarrow\) best minimal reductions on single test problems down to 12% and 18%
▶ But: Solving cut problems is expensive \(\rightarrow\) runtime is increased by cuts!
▶ Comparison: B& B based OA is preferable regarding running time.
Outlook

- Efficient cut computation
- Strategies to decide in which nodes cuts are generated
- Apply and test cut generation techniques currently used for non-convex MINLPs [Saxena, Bonami, Lee ’08]
- Test of integer rounding cuts [Atamtuerk, Narayanan ’07]
- ...
Thank you for your attention!