Fixed-Charge Transportation on a Path: LP formulations

RealOpt Seminar, Bordeaux, March 2011
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Agenda

- Multi-Item Production Planning by MIP (8 slides)
- Fixed-Charge Transportation Problem (6 slides)
- FCT on a Path
  - Sub/Super-modularity and Valid Inequalities (9 slides)
  - Extreme solutions and LP Extended Formulation (7 slides)
  - Projection (6 slides)
- Conclusion and Discussion (1 slide)
Big Bucket Lot Sizing

DATA

\[ D^i_t \] demand for item \( i \) in period \( t \)
\[ C_t \] total capacity in period \( t \)
\[ p^i_t, f^i_t \] unit and fixed cost of producing \( i \) in period \( t \)

VARIABLES

\[ x^i_t \] production of item \( i \) in period \( t \)
\[ y^i_t \] setup binary variable for item \( i \) in period \( t \)
Big Bucket Lot Sizing

\[ \min \sum_{i \in I} \sum_{t=1}^{T} (p_t^i x_t^i + f_t^i y_t^i), \]
\[ \sum_{t=1}^{k} x_t^i \geq \sum_{t=1}^{k} D_t^i \quad \forall i \in I, k = 1, \ldots, T - 1 \]
\[ \sum_{t=1}^{T} x_t^i = \sum_{t=1}^{T} D_t^i \quad \forall i \in I \]
\[ \sum_{i \in I} x_t^i \leq C_t \quad t = 1, \ldots, T \]
\[ 0 \leq x_t^i \leq \min(\sum_{k=t}^{T} D_k^i, C_t) y_t^i \quad \forall i \in I, t = 1, \ldots, T, \]
\[ y_t^i \in \{0, 1\}, \quad \forall i \in I, t = 1, \ldots, T. \]
Exponential explosion with size but capacity also!

\[\begin{align*}
\text{Capacity} & \quad \# \text{B&B nodes} \\
720 & \quad 700 \quad 680 \quad 660 \quad 640 \quad 620 \quad 600 \quad 580 \quad 570 \quad 560 \\
720 & \quad 700 \quad 680 \quad 660 \quad 640 \quad 620 \quad 600 \quad 580 \quad 570 \quad 560 \\
\end{align*}\]

- tr6-15 nodes
- tr6-30 nodes
- tr6-15 time
- tr6-30 time

Cplex 11.2
Formulations of a Mixed-Integer Set

- There are many formulations of the same mixed-integer set.
- A *mixed integer* set $Q$ is a subset of $\mathbb{Z}^n \times \mathbb{R}^p$.
- A *polyhedron* is a set described by linear inequalities.
- A polyhedron $P$ is a formulation of a mixed-integer set $X$ if
  \[ X = P \cap \{ \mathbb{Z}^n \times \mathbb{R}^p \} \]
- The “smaller” the polyhedron $P$, the better (“tighter”, “stronger”) the formulation.
- The best, ideal formulation is the convex hull of $X$: the is the “Graal” of integer programming.
  - The Graal is only attainable for easy (polynomially solvable) problems…
  - And even then, it typically includes an exponential number of constraints.
Valid Inequalities and Relaxations

• The inequality $\sum_{i=1}^{n} a_i x_i \leq b$ is valid for the set $Y$ if all points $x \in Y$ satisfy the inequality.

• $R$ is a relaxation of $Y$ if $Y \subseteq R$.

• If an inequality is valid for the relaxation $R$, it is also valid for the set $Y$. 
Extended Formulations

• Instead of describing the convex hull directly with linear inequalities, one might prefer to describe a higher dimensional polyhedron whose projection yields what we want.
  • This is called an “extended formulation”
• The size of the extended formulation (number of variables and constraints) might be much smaller than its projection
**BBLs: FL reformulation ~ each single-item relaxation is formulated as its convex hull**

- $z_{t,k}^i$ represents the amount of item $i$ produced in $t$ to satisfy demand in $k$ ($x_t^i$ is disaggregated as $x_t^i = \sum_k z_{t,k}^i$)

\[
\begin{align*}
\min & \sum_{i \in I} \sum_{t=1}^{T} \left( \sum_{k=t}^{T} p_t^i z_{t,k}^i + f_t^i y_t^i \right), \\
\sum_{t=1}^{k} z_{t,k}^i &= D_k^i & \forall i \in I, k = 1, \ldots, T, \\
\sum_{i \in I} \sum_{k=t}^{T} z_{t,k}^i &\leq C_t & t = 1, \ldots, T, \\
0 &\leq z_{t,k}^i \leq \min(C_t, D_k^i) y_t^i & \forall i \in I, t = 1, \ldots, T, k = t, \ldots, T, \\
y_t^i &\in \{0, 1\}, & \forall i \in I, t = 1, \ldots, T.
\end{align*}
\]
Substantial improvement, but not satisfactory…
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Fixed Charge Transportation problem

\[
\begin{align*}
\text{max} & \quad \sum_{i \in I} \sum_{j \in J} (p_{i,j} x_{i,j} - f_{i,j} y_{i,j}), \\
\sum_{j \in J} x_{i,j} & \leq C_i, \quad \forall i \in [1, m] \\
\sum_{i \in I} x_{i,j} & \leq D_j, \quad \forall j \in [1, n] \\
0 & \leq x_{i,j} \leq \min(C_i, D_j) y_{i,j}, \quad \forall i \in [1, m], j \in [1, n], \\
y & \in \{0, 1\}^{m \times n},
\end{align*}
\]
BBLS with demand only for t=T is a Fixed Charge Transportation prob.

\[
\begin{align*}
\min & \quad \sum_{i \in I} \sum_{t=2}^{T} \left( (p_t^i - p_1^i)x_t^i + f_t^i y_t^i \right) + \text{constant}, \\
\quad & \sum_{t=2}^{T} x_t^i \leq D_T^i \quad \forall i \in I, \\
\quad & \sum_{i \in I} x_t^i \leq C_t \quad t = 2, \ldots, T, \\
\quad & 0 \leq x_t^i \leq \min(D_T^i, C_t)y_t^i \quad \forall i \in I, t = 2, \ldots, T, \\
\quad & y_t^i \in \{0, 1\}, \quad \forall i \in I, t = 2, \ldots, T.
\end{align*}
\]
FCT as a special case of BBLS

Little hope to solve BBLS if one cannot solve FCT
FCT as relaxation of BBLS

\[
\min \sum_{i \in I} \sum_{t=1}^{T} \left( \sum_{k=t}^{T} \left( (p^i_t - M) z^i_{t,k} + f^i_{t} y^i_{t,T} \right) \right),
\]

\[
\sum_{t=1}^{k} z^i_{t,k} \leq D^i_k \quad \forall i \in I, k = 1, \ldots, T,
\]

\[
\sum_{i \in I} \sum_{k=t}^{T} z^i_{t,k} \leq C_t \quad t = 1, \ldots, T,
\]

\[
0 \leq z^i_{t,k} \leq \min(C_t, D^i_k) y^i_{t,k}, \quad \forall i \in I, t = 1, \ldots, T, k = t, \ldots, T,
\]

\[
y^i_{t,k} \in \{0, 1\}, \quad \forall i \in I, t = 1, \ldots, T, k = t, \ldots, T,
\]

\[
y^i_{t,k} = y^i_{t,k+1} \quad \forall i \in I, t = 1, \ldots, T, k = t, \ldots, T - 1,
\]
FCT as relaxation of BBLS
FCT: Literature

• Fairly basic problem in management, but only little polyhedral work
  • if only one depot or client, then FCT reduces to single node flow set
    • NP-Hard
    • (lifted) flow cover and reverse flow cover inequalities (Padberg et al. ’85, van Roy and Wolsey 87, Gu et al. ’99)
  • if fixed cost on nodes, then Facility Location problem
    • NP-Hard
    • inequalities described are variants of flow cover inequalities (Aardal ’98, Carr et al. ’00)
• This work = study of FCT on paths (can be solved in polynomial time)
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FCT on a Path

\[ \begin{align*}
\text{max} & \quad \sum_{i=1}^{n} p_j x_j - \sum_{i=1}^{n} f_j y_j, \\
& \quad x_1 \leq a_0, \\
& \quad x_i + x_{i+1} \leq a_i, \quad \forall i \in [1, n-1], \\
& \quad x_n \leq a_n, \\
& \quad 0 \leq x_i \leq \min(a_{i-1}, a_i) y_i, \quad \forall i \in [1, n], \\
& \quad y \in \{0, 1\}^n,
\end{align*} \]

\((X^{FCTP})\)
Conforti, Di Summa, Eisenbrand, Wolsey ’09 framework.

- Each node is a variable (round=continuous, square=integer)
- Each arc (i,j) is a constraint of the form $x_j - x_i \leq a_{i,j}$
FCTP in C,DS,E,W ’09 framework.

- Our work:

- Di Summa, Wolsey ’10:
FCTP in C,DS,E,W ’09 framework.

- Our work:

- Di Summa, Wolsey ’10:
Valid Inequalities

- Let $\varphi(S)$ be the maximum that can be flowed from the depots to the clients when the edges in $S$ are open and the other closed

$$\varphi(S) = \max \sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j},$$

$$\sum_{j \in J} x_{i,j} \leq C_i, \quad \forall i \in [1, m]$$

$$\sum_{i \in I} x_{i,j} \leq D_j, \quad \forall j \in [1, n]$$

$$x_{i,j} \geq 0, \quad \forall i \in [1, m], j \in [1, n],$$

$$x_{i,j} = 0, \quad \forall (i, j) \notin S$$

Let $\rho_i(S) = \varphi(S+i) - \varphi(S) =$ gain in opening $i$ when $S$ is open

- $\varphi(S)$ is submodular if $\rho_i(S) \geq \rho_i(T)$ for $S \subseteq T$

- $\varphi(S)$ is supermodular if $\rho_i(S) \leq \rho_i(T)$ for $S \subseteq T$
Single node flow set = one factory & n clients

the more is already open the less the gain from opening is important

= The function $\varphi$ for single-node flow set is submodular
Single-Node Flow Set

- The single-node flow set is the special case where there is only one depot and n clients.

\[
\sum_{j=1}^{n} x_j \leq C, \\
0 \leq x_j \leq D_j y_j, \quad \forall j \in [1, n] \\
y_j \in \{0, 1\}, \quad \forall j \in [1, n]
\]

- The flow cover inequality is a valid inequality for this set.

\[
\sum_{j} x_j \leq C - \sum_{j} \rho_j (N - j)(1 - y_j) \\
x_1 + x_2 + x_3 \leq 10 - 2(1 - y_1) - 1(1 - y_2) - 0(1 - y_3)
\]
FCT on a Path

\[
\max \quad \sum_{i=1}^{n} p_j x_j - \sum_{i=1}^{n} f_j y_j, \\
X_{\text{FCTP}}
\]

\[
x_1 \leq a_0, \\
x_i + x_{i+1} \leq a_i, \quad \forall i \in [1, n-1], \\
x_n \leq a_n, \\
0 \leq x_i \leq \min(a_{i-1}, a_i) y_i, \quad \forall i \in [1, n], \\
y \in \{0, 1\}^n,
\]
How much more can I transport when opening a new connection between factory & client?

Sometimes “positive externality”: the more is already open the more the gain from opening is important

= The function $\varphi$ for paths is neither sub- nor supermodular
but we can still characterize this

**Theorem.** Let $S \subset T \subseteq N$ and $i \in N \setminus T$ be given.
(a) If all elements of $T \setminus S$ and $i$ have the same parity, then $\rho_i(T) \geq \rho_i(S)$.
(b) If all elements of $T \setminus S$ and $i$ have opposite parity, then $\rho_i(T) \leq \rho_i(S)$. 
but we can still characterize this

Theorem. Let $S \subset T \subseteq N$ and $i \in N \setminus T$ be given.
(a) If all elements of $T \setminus S$ and $i$ have the same parity, then $\rho_i(T) \geq \rho_i(S)$.
(b) If all elements of $T \setminus S$ and $i$ have opposite parity, then $\rho_i(T) \leq \rho_i(S)$.

Path-modular inequalities Let $L$ be a subinterval of $N$, let $L = O_L \cup E_L$ be the partition of $L$ into odd and even numbers, and let $(j_1, j_2, \ldots, j_{|L|})$ be an ordering of $L$. Let $O_L^{j_k} = \{j_1, \ldots, j_k\} \cap O_L$ and let $E_L^{j_k} = \{j_1, \ldots, j_{k-1}\} \cap E_L$. We call the the following relation the path-modular inequality:

$$\sum_{i \in L} x_i \leq \phi(O_L) - \sum_{i \in O_L} \rho_i(O_L \cup E_L^i \setminus O_L^i)(1 - y_i) + \sum_{i \in E_L} \rho_i(O_L \cup E_L^i \setminus O_L^i)y_i$$
Examples of Path-Modular inequalities

Let $L = N = [1, 6]$ and the ordering be $(2, 4, 6, 1, 3, 5)$. Then the path-modular inequality is

$$
\sum_{i=1}^{6} x_i \leq \phi(\{1, 3, 5\}) + \rho_2(\{1, 3, 5\})y_2 + \rho_4(\{1, 2, 3, 5\})y_4 + \rho_6(\{1, 2, 3, 4, 5\})y_6

- \rho_1(\{2, 3, 4, 5, 6\})(1 - y_1) - \rho_3(\{2, 4, 5, 6\})(1 - y_3) - \rho_5(\{2, 4, 6\})(1 - y_5)

\leq 16 + 3y_2 + 0y_4 + 0y_6 - 3(1 - y_1) - 0(1 - y_3) - 2(1 - y_5)

= 11 + 3y_1 + 2y_5 + 3y_2
$$

and is facet-defining. With the different ordering $(4, 3, 6, 2, 1, 5)$, one obtains

$$
\sum_{i=1}^{6} x_i \leq \phi(\{1, 3, 5\}) + \rho_4(\{1, 3, 5\})y_4 - \rho_3(\{1, 4, 5\})(1 - y_3) + \rho_6(\{1, 4, 5\})y_6

+ \rho_2(\{1, 4, 5, 6\})y_2 - \rho_1(\{2, 4, 5, 6\})(1 - y_1) - \rho_5(\{2, 4, 6\})(1 - y_5)

\leq 16 + 0y_4 - 4(1 - y_3) + 3y_6 + 3y_2 - 2(1 - y_1) - 2(1 - y_5)

= 8 + 2y_1 + 3y_2 + 4y_3 + 0y_4 + 2y_5 + 3y_6
$$
Formulations of a Mixed-Integer Set

- The “smaller” the polyhedron $P$, the better (“tighter”, “stronger”) the formulation
- Have we reached the Graal?
  - Is the polyhedron described by the path-modular inequalities the convex hull of the feasible solutions to FCTP?
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Extreme solutions

**Definition.** For given $j \in [0, n + 1]$, let $\bar{\alpha}_{i,j}$ for $i \in [0, n + 1]$ be the unique solution of the following system of $n + 2$ linear equations:

$$x_i + x_{i+1} = a_i \text{ for } i \in [0, n] \text{ and } x_j = 0.$$

**Lemma.** Let $x$ be an extreme point of $X^{FCTP}$. Then $x_i$ takes its value in the set $\{\bar{\alpha}_{i,j}\}_{j=0}^{n+1}$. 
Extreme solutions

Example. Consider the example with \( a = [5, 8, 6, 5, 7, 6] \).
Extreme solutions

**Example.** Consider the example with \( a = [5, 8, 6, 5, 7, 6] \).

\[
\bar{\alpha} = \begin{pmatrix}
0 & 8 & 2 & 7 & 0 & 6 & 3 \\
5 & 0 & 8 & 2 & 7 & 0 & 6 & 3 \\
3 & 8 & 0 & 6 & 1 & 8 & 2 & 5 \\
3 & -2 & 6 & 0 & 5 & -2 & 4 & 1 \\
2 & 7 & -1 & 5 & 0 & 7 & 1 & 4 \\
5 & 0 & 8 & 2 & 7 & 0 & 6 & 3 \\
1 & 6 & -2 & 4 & -1 & 6 & 0 & 3 \\
1 & 6 & -2 & 4 & -1 & 6 & 0 & 3 & 0
\end{pmatrix}
\]
Reshuffling $\alpha$...

\[
\bar{\alpha} = \begin{pmatrix}
0 & 5 & 0 & 8 & 2 & 7 & 0 & 6 & 3 \\
3 & 8 & 0 & 6 & 1 & 8 & 2 & 5 \\
3 & -2 & 6 & 0 & 5 & -2 & 4 & 1 \\
2 & 7 & -1 & 5 & 0 & 7 & 1 & 4 \\
5 & 0 & 8 & 2 & 7 & 0 & 6 & 3 \\
1 & 6 & -2 & 4 & -1 & 6 & 0 & 3 & 0 \\
\end{pmatrix}
\]

sort cols

\[
\begin{pmatrix}
0 & 0 & 2 & 3 & 5 & 6 & 7 & 8 \\
8 & 8 & 6 & 5 & 3 & 2 & 1 & 0 \\
-2 & -2 & 0 & 1 & 3 & 4 & 5 & 6 \\
7 & 7 & 5 & 4 & 2 & 1 & 0 & -1 \\
0 & 0 & 2 & 3 & 5 & 6 & 7 & 8 \\
6 & 6 & 4 & 3 & 1 & 0 & -1 & -2 \\
\end{pmatrix}
\]

\[
\beta = \begin{pmatrix}
0 & 2 & 3 & 5 & 6 & 7 & 8 \\
8 & 6 & 5 & 3 & 2 & 1 & 0 \\
-2 & 0 & 1 & 3 & 4 & 5 & 6 \\
7 & 5 & 4 & 2 & 1 & 0 & -1 \\
0 & 2 & 3 & 5 & 6 & 7 & 8 \\
6 & 4 & 3 & 1 & 0 & -1 & -2 \\
\end{pmatrix}
\]

shift rows

\[
\begin{pmatrix}
0 & 2 & 3 & 5 & 6 & 7 & 8 \\
8 & 6 & 5 & 3 & 2 & 1 & 0 \\
-2 & 0 & 1 & 3 & 4 & 5 & 6 \\
7 & 5 & 4 & 2 & 1 & 0 & -1 \\
0 & 2 & 3 & 5 & 6 & 7 & 8 \\
6 & 4 & 3 & 1 & 0 & -1 & -2 \\
\end{pmatrix}
\]

\[
\gamma_j = \begin{cases} 
\beta_{i,j} - \beta_{i,j-1} & \text{if } i \text{ is odd}, \\
\beta_{i,j} - \beta_{i,j+1} & \text{if } i \text{ is even}
\end{cases} = [2, 1, 2, 1, 1, 1].
\]
...to get the following extended formulation $Y$

(intended meaning: $z_{i,j} = 1$ if $x_i$ is at least $\beta_{i,j}$, and $=0$ otherwise)

\[
\sum_{j=1 : \beta_{i,j} > 0}^{n} \gamma_j z_{i,j} = x_i \quad \forall i \in N,
\]

\[
z_{i,j} \leq y_i, \quad \forall i, j \in N : \beta_{i,j} > 0
\]

\[
z_{i,j} + z_{i+1,j} \leq 1, \quad \forall i < n, j
\]

\[
z_{i,j} = 0, \quad \forall i, j : \beta_{i,j} > \min(a_{i-1}, a_i)
\]

\[
z_{i,j} = 1, \quad \forall i, j : \beta_{i,j} < 0,
\]

\[
y_i \leq 1, \quad \forall i \in N,
\]

\[z \geq 0\]
Extended Formulations

• Instead of describing the convex hull directly with linear inequalities, one might prefer to describe a higher dimensional polyhedron whose projection yields what we want.
  • This is called an “extended formulation”
• The size of the extended formulation (number of variables and constraints) might be much smaller than its projection
Y is a LP extended formulation of $\text{conv}(X^{FCTP})$

(Specialization of a more general result of CWZ’09)

**Proposition.** For any $(x, y) \in \mathbb{R}^n \times \mathbb{Z}^n$, $(x, y) \in X^{FCTP}$ if and only if there exists $z$ such that $(x, y, z)$ is feasible in $Y$.

**Sketch of proof.** ⇒ For any extreme points $(x, y)$ of $X^{FCTP}$ take $z_{i,k} = 1$ if $x_i$ takes a value at least $\bar{\beta}_{i,k}$ and 0 otherwise.

⇐ Show that $x_{i-1} + x_i \leq a_i$ is linearly implied by inequalities of $Y$.

**Proposition.** Extreme points of $Y$ are integral.

**Sketch of proof.** The constraint matrix associated with $Y$ is (essentially) the dual of a network flow matrix.
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Projecting the extended formulation

Testing if a point \((x^*, y^*)\) satisfying \(y^* \leq 1\) belongs to \(\text{conv}(X^{FCTP})\) is equivalent to testing whether the following LP in variables \(z\) is feasible:

\[
\begin{align*}
\text{max} & \quad 0, \\
\sum_{k \in K_i} \gamma_k z_{i,k} &= x_i^* \quad \forall i \in N, \\
z_{i,k} &\leq y_i^* \quad \forall i \in N, k \in K_i, \\
z_{i,k} + z_{i+1,k} &\leq 1, \quad \forall i \in [1, n-1], k \in K_i \cap K_{i+1}, \\
z_{i,k} &\geq 0, \quad \forall i \in N, k \in K_i,
\end{align*}
\]

where \(K_i = \{k \in [1, \bar{m}] : 0 < \bar{\beta}_{i,k} \leq \min(a_{i-1}, a_i)\}\). Through LP duality, this is equivalent to testing whether the following LP is bounded:

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^n x_i^* \Delta_i + \sum_{i=1}^n \sum_{k \in K_i} y_i^* \delta_{i,k} + \sum_{i=1}^{n-1} \sum_{k \in K_i \cap K_{i+1}} \rho_{i,k}, \\
\gamma_k \Delta_i + \delta_{i,k} + \rho_{i-1,k} + \rho_{i,k} &\geq 0, \quad \forall i \in N, k \in K_i \\
\rho_{i,k} & = 0, \quad \forall i \notin [1, n-1] \text{ or } k \notin K_i \cap K_{i+1}, \\
\delta, \rho &\geq 0,
\end{align*}
\]
Projecting the extended formulation

\[
\begin{align*}
\min & \sum_i x_i^* \Delta_i + \sum_{i,j} y_i^* \delta_{i,j} + \sum_{i,j} \rho_{i,j} \\
\gamma_j \Delta_i + \delta_{i,j} + \rho_{i-1,j} + \rho_{i,j} & \geq 0, \quad \forall i > 1, j \\
\delta, \rho & \geq 0
\end{align*}
\]

rescale \( \delta_{i,j}, \rho_j \) by \( \gamma_j \)

\[
\begin{align*}
\min & \sum_i x_i^* \Delta_i + \sum_{i,j} (y_i^* \gamma_j) \delta_{i,j} + \sum_{i,j} (\rho_{i,j} \gamma_j) \\
\Delta_i + \delta_{i,j} + \rho_{i-1,j} + \rho_{i,j} & \geq 0, \quad \forall i > 1, j, \quad (1) \\
\delta, \rho & \geq 0
\end{align*}
\]

Proposition. Constraint matrix associated to (1) is TU.
Proof that projection cone constraint matrix is TU

\[ \Delta_i \quad \rho_i \]

\begin{align*}
+ &+ \\
+ &- \\
+ &- \\
0 &+ \\
0 &- \\
0 &+ \\
- &+ \\
- &+ \\
+ &+ \\
\end{align*}
Projecting the extended formulation

\[
\begin{align*}
\min \sum_i x_i^* \Delta_i + \sum_{i,j} (y_i^* \gamma_j) \delta_{i,j} + \sum_{i,j} (\rho_{i,j} \gamma_j) \\
\Delta_i + \delta_{i,j} + \rho_{i-1,j} + \rho_{i,j} & \geq 0, \quad \forall i > 1, j, \\
\delta, \rho & \geq 0
\end{align*}
\]

• Interesting rays satisfy \(\Delta \leq 0\) and \(\Delta_i < 0\) for some \(i\).

• Normalize rays with \(\Delta \geq -1 \rightarrow \Delta_i = 0\) or \(-1\) (because matrix is TU).

• Interesting extreme rays are of the form \([0,0,0,-1,-1,-1,0,0]\)
  • otherwise it is the sum of two other rays.
  • that’s only \(O(n^2)\) distinct possible values for \(\Delta\).
Projecting the extended formulation

- For $\Delta$ fixed, the problem is then separable in $j$. For each $j$, the remaining problem is of the form

$$\min \sum_{i=1}^{n} y_i^* \delta_i + \sum_{i=1}^{n-1} \rho_i,$$

$$\delta_1 + \rho_1 \geq 1,$$

$$\delta_i + \rho_{i-1} + \rho_i \geq 1, \quad \forall i \in [2, n - 1],$$

$$\delta_n + \rho_{n-1} \geq 1,$$

$$\delta, \rho \geq 0,$$
Separation: Assignment problem for each $j$

- matching problem in a bipartite graph.
- optimal solution depends only of the ordering of

$$\tilde{y}_i^* =\begin{cases} y_i^* & \text{if } i \text{ is even}, \\ 1 - y_i^* & \text{if } i \text{ is odd}. \end{cases}$$

- This is independent on $j$
- The following set of rays/valid inequalities is sufficient to describe $\text{conv}(X_{\text{FCTP}})$

$$\sum_{i=l}^{l'} x_i \leq \tau(\mathcal{L}) + \sum_{i=l}^{l'} \sigma(i, \mathcal{L}) y_i,$$

for all $[l, l'] \in [1, n]$ and permutations $\mathcal{L}$ of $[l, l']$
Path-modular inequalities are sufficient

**Theorem** Together with trivial bounds on variables, the path-modular inequalities are sufficient to describe the convex hull of $X^{FCTP}$.

**Idea of proof.** For both families, each inequality is defined by an interval $[l, l']$ and a permutation $\mathcal{L}$ of this interval. Use separation algorithms for both families to show that inequalities corresponding to the same interval and permutation are tight at the same extreme points of $X^{FCTP}$. 
Agenda

• Multi-Item Production Planning by MIP (8 slides)
• Fixed-Charge Transportation Problem (6 slides)
• FCT on a Path
  • Sub/Super-modularity and Valid Inequalities (9 slides)
  • Extreme solutions and LP Extended Formulation (7 slides)
  • Projection (6 slides)
• Conclusion and Discussion (1 slide)
Conclusion

• Analysis of FCTP is a first step in a research program to better solve FCT:
  • generalize inequalities to even cycle, but also trees? comets?
  • Lift additional arcs into path-modular inequality?
  • How to find violated path-modular inequalities from fractional solutions of FCT?

• Implement this into a software and do computational experiments
  • fine-tuning of some parameters is typically necessary
  • this is the ultimate proof of usefulness
Appendices
Computational experiments

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XPRESS-MP 2007, laptop
Facets of a small instance of FCT

- We enumerated all facets of an 3x2 FCT instance:
  - 136 facets,
  - 19 are trivial
  - 7 are flow cover (or path-modular inequalities of length 2)
  - 6 are facets of a single-node flow relaxation of size 3 (flow cover, lifted flow cover or other inequalities).
  - 8 path-modular inequalities of size 3,
  - 3 facets are path-modular inequalities of size 4
  - 4 facets are even-cycle modular inequalities of size 4.
  - the other 89 facets are unexplained
Extreme Solutions: all capacities tight except one within reg. interval
Extreme Solutions: all capacities tight except one within reg. interval

$$\alpha_{i,j} = \begin{cases} 
\min(a_{i-1}, a_i) & \text{if } j = i, \\
\min(a_{i-1} - \alpha_{i-1,j}, a_i) & \text{if } j < i, \\
\min(a_i - \alpha_{i+1,j}, a_{i-1}) & \text{if } j > i.
\end{cases}$$

$$\bar{\alpha}_{i,j} = \begin{cases} 
0 & \text{if } j = i, \\
a_{i-1} - \bar{\alpha}_{i-1,j} & \text{if } j < i, \\
a_i - \bar{\alpha}_{i+1,j} & \text{if } j > i.
\end{cases}$$

$$\alpha = \begin{pmatrix}
5 & 2 & 5 & 2 & 5 & 3 \\
3 & 6 & 1 & 6 & 2 & 5 \\
3 & 0 & 5 & 0 & 4 & 1 \\
2 & 5 & 0 & 5 & 1 & 4 \\
5 & 2 & 6 & 2 & 6 & 3 \\
1 & 3 & 0 & 3 & 0 & 3
\end{pmatrix}$$

$$\bar{\alpha} = \begin{pmatrix}
0 & 5 & 0 & 8 & 2 & 7 & 0 & 6 & 3 \\
3 & 8 & 0 & 6 & 1 & 8 & 2 & 5 \\
3 & -2 & 6 & 0 & 5 & -2 & 4 & 1 \\
2 & 7 & -1 & 5 & 0 & 7 & 1 & 4 \\
5 & 0 & 8 & 2 & 7 & 0 & 6 & 3 \\
1 & 6 & -2 & 4 & -1 & 6 & 0 & 3 \\
1 & 6 & -2 & 4 & -1 & 6 & 0 & 3 & 0
\end{pmatrix}$$
Structure of possible values of $x_i$ at extreme points

$$
\alpha_{i,j} = \begin{cases} 
\min(a_{i-1}, a_i) & \text{if } j = i, \\
\min(a_{i-1} - \alpha_{i-1,j}, a_i) & \text{if } j < i, \\
\min(a_i - \alpha_{i+1,j}, a_{i-1}) & \text{if } j > i.
\end{cases}
$$

Proposition.

$$
\begin{align*}
\alpha_{i,i} &\geq \alpha_{i,i-2} \geq \ldots \geq \alpha_{i,1} \text{ or } 2 \geq \alpha_{i,2} \text{ or } 1 \geq \ldots \geq \alpha_{i,i-3} \geq \alpha_{i,i-1}, \\
\alpha_{i,i} &\geq \alpha_{i,i+2} \geq \ldots \geq \alpha_{i,n-1} \text{ or } n \geq \alpha_{i,n} \text{ or } n-1 \geq \ldots \geq \alpha_{i,i+3} \geq \alpha_{i,i+1}.
\end{align*}
$$
Structure of possible values of $x_i$ at extreme points

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\end{cases}
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Proposition.

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\alpha_{i,i} \geq \alpha_{i,i-2} \geq \ldots \geq \alpha_{i,1} \text{ or } 2 \geq \alpha_{i,2} \text{ or } 1 \geq \ldots \geq \alpha_{i,i-3} \geq \alpha_{i,i-1},
\]

\[
\alpha_{i,i} \geq \alpha_{i,i+2} \geq \ldots \geq \alpha_{i,n-1} \text{ or } n \geq \alpha_{i,n} \text{ or } n-1 \geq \ldots \geq \alpha_{i,i+3} \geq \alpha_{i,i+1}.
\]
Regeneration Intervals

• Each regeneration interval is characterized by
  • i: its start node
  • j-1: its end node
  • k: a node where the capacity $a_k$ is not binding

• Each interval $[i, k, j]$ is essentially the concatenation of two half-intervals $[i, k]$ and $[k, j]$

• Two half intervals $[i, k]$ and $[k, j]$ form a regeneration interval if $\alpha_{k-1, i} + \alpha_{k, j} \leq a_k$