2d-Packing: Beginning of formulation
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OUTLINE:

1. Hypothesis and Definition for 2d PP [17, 18]
2. Interval Graph Representation : Packing Classes
3. Graph theory
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Hypothesis and Definition for 2d PP

The Packing problems satisfies the following constraints:

1. Orthogonality
2. Closedness
3. Disjointness
4. Fixed Orientations

Definition 1 (Packing) A feasible packing: orthogonality, closedness, disjointness, fixed orientation

function \( p : i \in V \rightarrow (x, y) \) such that

\[
\begin{align*}
    x_i + w_i & \leq W \quad \forall i \in V \\
    y_i + h_i & \leq H \quad \forall i \in V
\end{align*}
\]

\( \forall (i, j) \in V \times V, \exists d : I_i^d \cap I_j^d = \emptyset \)

where \( I_i^1 = [x_i, x_i + w_i) \) and \( I_i^2 = [y_i, y_i + h_i) \).

Drawback: symetries.

\[ \Rightarrow \] one defines \( x_{ij} = 1 \) if item \( i \) is on the side of \( j \) according to X-axis.
And one defines \( y_{ij} = 1 \) if item \( i \) is on the side of \( j \) according to Y-axis.
Interval Graph Representation: Packing Classes

Fixing items in parallel of already packed items along at least one dimension (i.e. fixing edge and non-edges). [17]

Principle:

- two items are processed in parallel (don’t share ressource/dim. \(d\)), iff their projection on the axis \(X_d\) don’t intersect each other.

- A non-edge in \(G_d\), i.e. an edge in \(G_d^{c}\), implies the 2 items share ressource \(d\) and hence must be processed sequencially.

Properties:

P1: \(G_d\) comparability graph, \(G_d^{c}\) interval graph: the graph \(G_d^{c}\) must be an interval graph.

P2: closedness: \(\sum_{i \in C} w_i^d \leq W^d\) \(\forall\) clique \(C \in G_d\) \(\Leftrightarrow\) each clique \(C\) of \(G_d\) is \(x_d\) feasible. (a clique represents a set of boxes that are placed in sequence along dimension \(d\))

P3: disjointness: \(\forall (i, j), \exists d: (i, j) \in E_d\), i.e. \(\cup_d E_d = E\) (2 items must be in sequence along a least one dimension).
Graph theory

Interval graph:

Clique and Stable set:
Example: $S = \{b, c, d\}$ is a stable set and $S = \{a, d, e\}$ is a clique.

Definition 2 (Transitive orientation) An directed graph $D = (V, A)$ such that $(a, b) \in A \cap (b, c) \in A \Rightarrow (a, c) \in A$

Comparability graph:

Lemna: the complement of a interval graph is a comparability graph.
Packing classes: A $d$-uples $E = (E_1, \ldots, E_d)$ of sets of (edge sets of) graphs satisfying $P_1, P_2, P_3$ is a packing class.

⇒ Purpose: Enumerate packing classes / decide whether there are packing classes; instead of enumerate feasible packing.
Formulation

Theorem 1 (Ghouilá-Houri 1962) A graph is a comparability graph iff it doesn’t contain a 2-chordless cycle of odd length.

Theorem 2 (Gilmore and Hoffman 1964) A cocomparability graph (comparability graph complement) is an interval graph iff it doesn’t contain the chordless cycle $C_4$ of length 4 as an induced subgraph.

Theorem 3 (Gilmore and Hoffman 1964) A graph is an interval graph if and only if its maximal cliques can be linearly ordered in such a way that for every vertex in the graph the maximal cliques to which it belongs occur consecutively in the linear order.

Theorem 4 (Olariu 1991) A graph $G = (V, E)$ is an interval graph if and only if there exists a linear order $<$ on the set of its vertices such that for every choice of vertices $u, v, w$ with $u < v$ and $v < w$,

$$(u, w) \in E \text{ implies } (u, v) \in E$$

(1)
Variables: Let,
$z_{ij}^d$ edge set boolean variable ($z_{ij}^d = 1$ iff the edge $(i, j)$ in $G$).
$z_{ij}^d = 1 - x_{ij}^d$
$r_{ij}^d$ linear order boolean variables ($r_{ij}^d = 1$ iff $i < j$ w.r.t. order $<$)

Property P1 formulation:
$\forall d \in \{1, \ldots, D\}; \forall i, j \in V,$

\begin{align*}
    r_{ij}^d + r_{jk}^d + z_{ik}^d & \leq z_{ij}^d + 2 \\
    r_{ij}^d + r_{ji}^d & = 1 \\
    r_{ij}^d + r_{jk}^d & \leq r_{ik}^d + 1
\end{align*}

Comments: In. (2) is In. (1), In. (3, 4) are linear order axioms

Property P2 formulation:
Actually, I don’t know to formulate this constraint. In fact, it’s difficult to repear the set of stable subset. However, It’s possible to use an cut generation or generate all constraint before which exceeded the bin size to solve the OPP problem.

Property P3 formulation:
$\forall i, j \in V,$

\[ \forall i, j \sum_{d=1}^{D} z_{ik}^d \leq |D| - 1 \]

Problem: If it exist several order, one have symetries in the formulation.
Idea for P2:
One can prove that the linear order in $\mathcal{G}$ is an transitive orientation in $G$ (i.e $i < j \Rightarrow (i,j) \in A$). The reverse is true if one used a process based on the topology order.

Let,
$a_{ij}^d$ arc set boolean variable ($a_{ij}^d = 1$ iff the arc $(i,j)$ exist in $G_d$).
$l_i^d$ positive real variables which represents the value of longest path problem.

Arc definition in $G_d$:
$\forall d \in \{1, \ldots, D\}; \forall i, j \in V,$

$$a_{ij}^d \leq r_{ij}^d \quad (5)$$
$$a_{ij}^d + a_{ji}^d \geq 1 - z_{ij}^d \quad i < j \quad (6)$$
$$a_{ij}^d \leq 1 - z_{ij}^d \quad (7)$$

Longest path value in $G_d$:
$\forall d \in \{1, \ldots, D\}; \forall i, j \in V,$

$$l_{ij}^d \geq l_i^d a_{ij}^d + w_j^d a_{ij}^d \quad (8)$$
$$l_i^d \leq W^d \quad (9)$$

Remark:
linearization of the constraint 8 may be:

$$a_j^d \geq a_i^d + w_j^d a_{ij}^d - (W^d + 1)(1 - a_{ij}^d)$$
Bibliography


